An Exact Equation for the Median Return

John Norstad
j-norstad@northwestern.edu
http://www.norstad.org

April 23, 2004
Updated: November 3, 2011

Abstract

A short derivation of the exact equation for computing the median or “annualized” return of a lognormal random walk asset, given the simply compounded average or “expected” return and standard deviation of the simply compounded returns.
1 An Exact Equation for the Median Return

Let $Y$ be a lognormal random variable giving the ending value after one year of a $1\text{ investment in an asset. Let:}$

$$\mu = \text{the expected continuously compounded return of the asset.}$$
$$\sigma = \text{the standard deviation of the continuously compounded returns.}$$

By Proposition 5 in reference [1] we have:

$$E(Y) = e^{\mu + \frac{1}{2}\sigma^2}$$
$$\text{Var}(Y) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$$

Let:

$$r = \text{the simply compounded expected return of the asset.}$$
$$s = \text{the standard deviation of the simply compounded returns.}$$

Then:

$$1 + r = E(Y) = e^{\mu + \frac{1}{2}\sigma^2}$$
$$s^2 = \text{Var}(Y) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$$

We can solve these equations for $\sigma^2$ and $\mu$:

$$\sigma^2 = \log \left( \frac{s^2}{(1 + r)^2} + 1 \right)$$
$$\mu = \log(1 + r) - \frac{1}{2}\sigma^2$$

$$= \log(1 + r) - \frac{1}{2} \log \left( \frac{s^2}{(1 + r)^2} + 1 \right)$$
$$= \log(1 + r) - \log \left( \sqrt{\frac{s^2}{(1 + r)^2} + 1} \right)$$
$$= \log \left( \frac{1 + r}{\sqrt{\frac{s^2}{(1 + r)^2} + 1}} \right)$$
$$= \log \left( \frac{(1 + r)^2}{\sqrt{s^2 + (1 + r)^2}} \right)$$
Let: 

\[ a = \text{the median yearly simply compounded return} \]

Then:

\[
\begin{align*}
    a &= e^{\mu} - 1 \\
    &= e\log\left(\frac{(1+r)^2}{\sqrt{s^2 + (1+r)^2}}\right) - 1 \\
    &= \frac{(1+r)^2}{\sqrt{s^2 + (1+r)^2}} - 1
\end{align*}
\]

This is our exact equation for the median return.

If we take the Taylor series expansion of this equation about \( s = 0 \), the first three terms give the following approximation:

\[
a \approx r - \frac{\frac{1}{2} s^2}{1 + r}
\]

This is similar to, but a bit more accurate than, the frequently seen approximation:

\[
a \approx r - \frac{1}{2} s^2
\]
References